Lyapunov Exponents of Free Operators

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Abstract

Lyapunov exponents of a dynamical system are a useful tool to gauge the stability and complexity of the system. This paper offers a definition of Lyapunov exponents for a sequence of free linear operators. The definition is based on the concept of the extended Fuglede-Kadison determinant. We establish the existence of Lyapunov exponents, derive formulas for their calculation, and show that Lyapunov exponents of free variables are additive with respect to operator product. We illustrate these results using an example of free operators whose singular values are distributed by the Marchenko-Pastur law, and relate this example to C. M. Newman's "triangle" law for the distribution of Lyapunov exponents of large random matrices with independent Gaussian entries. As an interesting by-product of our results, we derive a relation between the extended Fuglede-Kadison determinant and Voiculescu's S-transform.

1 Introduction

Suppose that at each moment of time, t_i, a system is described by a state function $\varphi(t_i)$ and evolves according to the law $\varphi(t_{i+1}) = X_i \varphi(t_i)$, where X_i is a sequence of linear operators. One can ask how small changes in the initial position of the system are reflected in its long-term behavior. If operators X_i do not depend on time, $X_i = X$, then the long-term behavior depends to a large extent on the spectrum of the operator X. If operators X_i do depend on time but can be modelled as a stationary stochastic process, then the long-term behavior of the system depends to a large extent on so-called *Lyapunov exponents* of the process X_i.

The largest Lyapunov exponent of a sequence of random matrices was investigated in a pioneering paper [4] by Furstenberg and Kesten. This study

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was followed in [13] by Oseledec, who researched other Lyapunov exponents and finer aspects of the asymptotic behavior of matrix products. These investigations were greatly expanded and clarified by many other researchers. In particular, in [14] Ruelle developed a theory of Lyapunov exponents for random compact linear operators acting on a Hilbert space. Lyapunov exponents for random $N \times N$ matrices when $N \to \infty$ were studied in [2], [9], [10], and [7].

The goal of this paper is to investigate how the concept of Lyapunov exponents can be extended to the case of free linear operators. It was noted recently by Voiculescu ([17]) that the theory of free operators can be a natural asymptotic approximation for the theory of large random matrices. Moreover, it was noted that certain difficult calculations from the theory of large random matrices become significantly simpler if similar calculations are performed using free operators. For this reason it is interesting to study whether the concept of Lyapunov exponents is extendable to free operators, and what methods for calculation of Lyapunov exponents are available in this setting.

Free operators are not random in the traditional sense so the usual definition of Lyapunov exponents cannot be applied directly. Our definition of Lyapunov exponents is based on the observation that in the case of random matrices, the sum of logarithms of the k largest Lyapunov exponents equals the rate at which a random k-dimensional volume element grows asymptotically when we consecutively apply operators X_i.

In the case of free operators we employ the same idea. However, in this case we have to clarify how to measure the change in the "t-dimensional volume element" after we apply operators X_i. It turns out that we can measure this change by a suitable extension of the Fuglede-Kadison determinant. Given this extension, the definition proceeds as follows: Take a subspace of the Hilbert space, such that the corresponding projection is free from all X_i and has the dimension t relative to the given trace. Next, act on this subspace by the sequence of free operators X_i. Apply the determinant to measure how the "volume element" in this subspace changes under these linear transformations. Use the asymptotic growth in the determinant to define the Lyapunov exponent corresponding to the dimension t.

It turns out that the growth of the t-dimensional volume element is exponential with a rate which is a function of the dimension t. We call this rate the *integrated Lyapunov exponent*. It is an analogue of the sum of the k largest Lyapunov exponents in the finite-dimensional case. The derivative of this function is called the *marginal Lyapunov exponent*. Its value at a point t is an analogue of the k-th largest Lyapunov exponent.

Next, we relate the marginal Lyapunov exponent $f_X(t)$ to the Voiculescu S-transform of the random variable $X_i^* X_i$. The relationship is very simple:

$$f_X(t) = -(1/2)\log\left[S_{X^*X}(-t)\right]. \tag{1}$$

Using this formula, we prove that the marginal Lyapunov exponent is decreasing in t, and derive an expression for the largest Lyapunov exponent. Formula (1) also allows us to prove the additivity of the marginal Lyapunov exponent with respect to operator product: If X and Y are free, then $f_{XY}(t) = f_X(t) + f_Y(t)$.

As an example of application of formula (1), we calculate Lyapunov exponents for variables X_i such that $X_i^* X_i$ are distributed as Free Poisson variables with parameter λ. The case $\lambda = 1$ corresponds to the random matrix case considered by C. M. Newman in [9], and the results of this paper are in agreement with Newman's "triangle" law. In addition, our results regarding the largest Lyapunov exponent agree with the results regarding the norm of products of large random matrices in [2]. Finally, our formula for computation of Lyapunov exponents seems to be easier to apply than the non-linear integral transformation developed in [9].

An interesting by-product of our results is a relation between the extended Fuglede-Kadison determinant and the Voiculescu S-transform, which allows expressing each of them in terms of the other. In particular, if Y is a positive operator and if $\{P_t\}$ is a family of projections which are free of Y and such that $E(P_t) = t$, then

$$\log S_Y(-t) = -2\frac{d}{dt}\left[\log\det\left(\sqrt{Y}P_t\right)\right]. \tag{2}$$

and if X is bounded and invertible, then

$$\log\det(X) = -\frac{1}{2}\int_0^1 \log S_{X^*X}(-t)\,dt. \tag{3}$$

Calculations related to (2) and (3) were performed by Haagerup and Larsen in [5] in their investigation of the Brown measure of R-diagonal operators. The Brown measure of an operator X is closely related to the determinant of $X - zI$, and Haagerup and Larsen computed the Brown measure of an R-diagonal operator X in terms of the S-transform of X^*X. However, it appears that formulas (2) and (3) have not been stated explicitly in [5].

In addition, Sniady and Speicher showed in [15] that an R-diagonal X can be represented in the triangular form and that the spectra of the diagonal elements in this representation satisfy certain inequalities in terms of the S-transform of X^*X. Sniady and Speicher used their result to give a different proof of the results in [5]. It is likely that Sniady and Speicher's method can also be used for a different proof of formulas (2) and (3).

The rest of the paper is organized as follows: Section 2 describes the extension of the Fuglede-Kadison determinant that we use in this paper. Section 3 defines the Lyapunov exponents of free operators, proves an existence theorem, and derives a formula for the calculation of Lyapunov exponents. Section 4

computes the Lyapunov exponents for a particular example. Section 5 connects the marginal Lyapunov exponents and the S-transform, proves additivity and monotonicity of the marginal Lyapunov exponent, and derives a formula for the largest Lyapunov exponent. In addition, it derives a relation between the determinant and the S-transform. And Section 6 concludes.

2 A modification of the Fuglede-Kadison determinant

Let $\mathcal{A}$ be a finite von Neumann algebra and E be a trace in this algebra. Recall that if X is an element of $\mathcal{A}$ that has a bounded inverse, then the Fuglede-Kadison determinant ([3]) is defined by the following formula:

$$\det(X) = \exp \frac{1}{2} E \log(X^* X). \tag{4}$$

The most important property of the Fuglede-Kadison determinant is its multiplicativity:

$$\det(XY) = \det(X) \det(Y). \tag{5}$$

This determinant cannot be extended (non-trivially) to operators with non-zero kernel if we require that property (5) holds for all X and Y.

However, if we do not insist on this property, then we can define an *extended determinant* as follows: Let $\log^{+\lambda}(t) =: \log t$ if $t > \lambda$ and $=: 0$ if $t \le \lambda$. Note that $E \log^{+\lambda}(X^* X)$ is a (weakly) decreasing function of λ on the interval $(0, 1)$, and therefore it converges to a limit (possibly infinite) as $\lambda \to 0$.

Definition 1

$$\det(X) = \exp \frac{1}{2} \lim_{\lambda \downarrow 0} E \log^{+\lambda}(X^* X).$$

This extension of the Fuglede-Kadison determinant coincides with the extension introduced in Section 3.2 of [8] .

Example 1 *Zero operator*

From Definition 1, if $X = 0$, then $\det X = 1$.

Example 2 *Finite dimensional operator*

Consider the algebra of n-by-n matrices $M_n(C)$ with the trace given as the normalization of the usual matrix trace: $E(X) = n^{-1} \operatorname{Tr}(X)$. Then the original Fuglede-Kadison determinant is defined for all full-rank matrices and equals the product of the singular values of the operator in the power of $1/n$. It is easy to see that this equals the absolute value of the usual matrix determinant in

the power of $1/n$. The extended Fuglede-Kadison determinant is defined for all matrices, including the matrices of rank $k < n$, and equals the product of non-zero singular values in the power of $1/n$.

We can write the definition of the determinant in a slightly different form. Recall that for a self-adjoint operator $X \in \mathcal{A}$ we can define its *spectral probability measure* as follows:

$$\mu_X(S) = E(1_S(X)),$$

where S is an arbitrary Borel-measurable set and 1_S is its indicator function. Then, the determinant of operator X can be written as

$$\det(X) = \exp \frac{1}{2} \lim_{\lambda \downarrow 0} \int_{\mathbb{R}^+} \log^{+\lambda}(t)\, \mu_{X^*X}(dt).$$

For all invertible X the extended determinant defines the same object as the usual Fuglede-Kadison determinant. For non-invertible X, the multiplicativity property sometimes fails. However, it holds if a certain condition on images and domains of the multiplicands is fulfilled:

Proposition 1 *Let V be the closure of the range of the operator X. If Y is an injective mapping on V and is the zero operator on $V^{\perp}$, then* $\det(YX) = \det(Y)\det(X)$.

The claim of this proposition is a direct consequence of Theorem 3.14 and Lemma 3.15(7) in [8].

3 Definition of Lyapunov exponents for free operators

A pair $(\mathcal{A}, E)$ is a *tracial W^*-non-commutative probability space* if $\mathcal{A}$ is a finite von Neumann algebra with a normal faithful tracial state E, and $E(I) = 1$. The trace E will be called the *expectation* by analogy with classical probability theory.

Let $\mathcal{A}_{1,}...,\mathcal{A}_n$ be sub-algebras of algebra $\mathcal{A}$, and let a_i be elements of these sub-algebras such that $a_i \in \mathcal{A}_{k(i)}$.

Definition 2 *The sub-algebras $\mathcal{A}_{1,}...,\mathcal{A}_n$ (and their elements) are called* free *or* freely independent *if $E(a_1...a_m) = 0$ whenever the following two conditions hold:*
(a) $E(a_i) = 0$ for every i, and
(b) $k(i) \neq k(i+1)$ for every $i < m$.

The random variables are called free or freely independent if the algebras that they generate are free. (See [18], [6], or [11] for more details on foundations of free probability theory.)

Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of free identically-distributed operators. Let $\Pi_n = X_n \dots X_1$, and let P_t be a projection which is free of all X_i and has the dimension t, i.e., $E(P_t) = t$.

Definition 3 *The* integrated Lyapunov exponent *corresponding to the sequence X_i is a real-valued function of $t \in [0,1]$ which is defined as follows:*

$$F(t) = \lim_{n \to \infty} \frac{1}{n} \log \det (\Pi_n P_t),$$

provided that the limit exists.

Remark: In the case of random matrices, Π_n is the product of independent identically-distributed random matrices. In this case, it turns out that the function defined analogously to $F(t)$ equals the sum of the tN largest Lyapunov exponents divided by N, where N is the dimension of the matrices and t belongs to the set $\{0/N, 1/N, \dots, N/N\}$.

Our first task is to prove the existence of the limit in the previous definition.

Theorem 1 *Suppose that X_i are free identically-distributed operators in a tracial W^*-probability space $\mathcal{A}$ with trace E. Let $u =: \dim \ker (X_i)$. Then*

$$F(t) = \begin{cases} \frac{1}{2} \lim_{\lambda \downarrow 0} E \log^{+\lambda} (P_t X_1^* X_1 P_t), & \textit{if } t \le 1-u, \\ \frac{1}{2} \lim_{\lambda \downarrow 0} E \log^{+\lambda} (P_u X_1^* X_1 P_u), & \textit{if } t > 1-u. \end{cases}$$

Before proving this theorem, let us make some remarks. First, this theorem shows that the integrated Lyapunov exponent of the sequence $\{X_i\}$ exists and depends only on the spectral distribution of $X_i^* X_i$.

Next, suppose that we know that $F(t)$ is differentiable almost everywhere. Then we can define the *marginal Lyapunov exponent* as $f(t) = F'(t)$. We can also define the *distribution function of Lyapunov exponents* by the formula: $\mathcal{F}(x) = \mu \{t \in [0,1] : f(t) \le x\}$, where μ is the usual Borel-Lebesgue measure. Intuitively, this function gives a measure of the set of the Lyapunov exponents which are less than a given threshold, x. In the finite-dimensional case it is simply the empirical distribution function of the Lyapunov exponents, i.e., the fraction of Lyapunov exponents that fall below the threshold x.

Proof of Theorem 1: The proof is through a sequence of lemmas. We will consider first the case of injective operators X_i and then will show how to generalize the argument to the case of arbitrary X_i.

Let P_A denote the projection on the closure of the range of operator A. In the following lemmas we always assume that operators belong to a tracial W^*-probability space $\mathcal{A}$ with trace E.

Lemma 1 *Suppose that operator A is injective, and that P_t is a projection of dimension t. Then projection P_{AP_t} is equivalent to P_t. In particular, $E(P_{AP_t}) = t$.*

Proof: Recall that polar decomposition is possible in $\mathcal{A}$. (See Proposition II.3.14 on p. 77 in [16] for details.) Therefore, we can write $AP_t = WB$, where W is a partial isometry and B is positive, and where both W and B belong to $\mathcal{A}$. By definition, the range of W is $[\mathrm{Range}(AP_t)]$, and the domain of W is $[x : Bx = 0]^{\perp} = [x : AP_t x = 0]^{\perp} = [x : P_t x = 0]^{\perp} = [\mathrm{Range}(P_t)]$. Therefore, P_{AP_t} is equivalent to P_t, with the equivalence given by the partial isometry W. In particular, $\dim(P_{AP_t}) = \dim(P_t)$, i.e., $E(P_{AP_t}) = t$. QED.

Lemma 2 *If A, A^*, and P_t are free from an operator subalgebra $\mathcal{B}$, then P_{AP_t} is free from $\mathcal{B}$.*

Proof: P_{AP_t} belongs to the W^*-algebra generated by I, A, A^*, and P_t. By assumption, this algebra is free from $\mathcal{B}$. Hence, P_{AP_t} is also free from $\mathcal{B}$. QED.

Let us use the notation $Q_k = P_{X_k \ldots X_1 P_t}$ for $k \geq 1$ and $Q_0 = P_t$. Then by Lemma 2, Q_k is free from X_{k+1}. Besides, if all X_i are injective, then their product is injective and, therefore, by Lemma 1, Q_k is equivalent to P_t.

Lemma 3 *If all X_i are injective, then*

$$\det(\Pi_n P_t) = \prod_{i=1}^{n} \det(X_i Q_{i-1}).$$

Proof: Note that $\Pi_n P_t = X_n Q_{n-1} X_{n-1} \ldots Q_1 X_1 Q_0$. We will proceed by induction. We need only to prove that

$$\det(X_{k+1} Q_k X_k \ldots Q_1 X_1 Q_0) = \det(X_{k+1} Q_k) \det(X_k \ldots Q_1 X_1 Q_0). \tag{6}$$

Let V_k be the closure of the range of $X_k \ldots Q_1 X_1 Q_0$. Since X_{k+1} is injective and Q_k is the projector on V_k, therefore $X_{k+1} Q_k$ is injective on V_k and equal to zero on $V_k^{\perp}$. Consequently, we can apply Proposition 1 and obtain (6). QED.

Now we are ready to prove Theorem 1 for the case of injective X_i. Using Lemma 3, we write

$$n^{-1} \log \det(\Pi_n P_t) = \frac{1}{n} \sum_{i=1}^{n} \log \det(X_i Q_{i-1}).$$

Note that X_i are identically distributed by assumption, Q_i have the same dimension by Lemma 1, and X_i and Q_{i-1} are free by Lemma 2. This implies that $\lim_{\lambda \downarrow 0} E \log^{+\lambda}(Q_{i-1} X_i^* X_i Q_{i-1})$ does not depend on i, and hence, $\det(X_i Q_{i-1})$ does not depend on i. Hence, using $i = 1$ we can write:

$$n^{-1} \log \det(\Pi_n P_t) = \log \det(X_1 P_t).$$

This finishes the proof for the case of injective X_i. For the case of non-injective X_i, i.e., for the case when $\dim \ker(X_i) > 0$, we need the following lemma.

Lemma 4 *Suppose that* P_t *is a projection operator free of* A *and such that* $E(P_t) = t$. *Then* $\dim\ker(AP_t) = \max\{1 - t, \dim\ker(A)\}$.

Proof: Let $V = (\operatorname{Ker} A)^{\perp}$ and let P_V be the projection on V. Then $E(P_V) = 1 - \dim\operatorname{Ker} A$. Note that $Ax = 0 \Longleftrightarrow P_V x = 0$. Consequently, $AP_t x = 0 \Longleftrightarrow P_V P_t x = 0$. Therefore, we have:

$$\begin{aligned}\dim\{x : AP_t x = 0\} &= \dim\{x : P_V P_t x = 0\} \\ &= \dim\{x : P_t P_V P_t x = 0\}.\end{aligned}$$

Since P_t and P_V are free, an explicit calculation of the distribution of $P_t P_V P_t$ shows that

$$\dim\{x : P_t P_V P_t x = 0\} = \max\{1 - t, 1 - \dim V\}.$$

QED.

Consider first the case when $0 < \dim\ker X_i \leq 1 - t$. This case is very similar to the case of injective X_i. Using Lemma 4 we conclude that $\dim\operatorname{Ker}(X_1 P_t) = 1 - t$, and therefore that $E(P_{X_1 P_t}) = t$. If, as before, we denote $P_{X_1 P_t}$ as Q_1, then the projection Q_1 is free from X_2, and $E(Q_1) = t$.

Similarly, we obtain that $E(P_{X_2 Q_1}) = t$. Proceeding inductively, we define $Q_k = P_{X_k Q_{k-1}}$ and conclude that Q_k is free from X_{k+1} and that $E(Q_k) = t$.

Next, we write $X_k ... X_1 P_t = X_k Q_{k-1} X_{k-1} Q_{k-2} ... X_1 Q_0$, where Q_0 denotes P_t, and note that $X_k Q_{k-1}$ is injective on the range of Q_{k-1}. Indeed, if it were not injective, then we would have $\dim(\operatorname{Ker}(X_k) \cap \operatorname{Range}(Q_{k-1})) > 0$. But this would imply that $\dim(\operatorname{Ker} X_k Q_{k-1}) > \dim(\operatorname{Ker} Q_{k-1}) = 1 - t$, which contradicts the fact that $\dim(\operatorname{Ker} X_k Q_{k-1}) = 1 - t$. Therefore, Proposition 1 is applicable and

$$\begin{aligned}\det(X_k ... X_1 P_t) &= \det(X_k Q_{k-1}) ... \det(X_1 Q_0) \\ &= [\det(X_1 P_t)]^k.\end{aligned}$$

Now let us turn to the case when $\dim\ker X_i = u > 1 - t$. Then $\dim\operatorname{Ker}(X_1 P_t) = 1 - u$ and therefore $E(P_{X_1 P_t}) = u$. Proceeding as before, we conclude that $E(Q_k) = u$ for all $k \geq 1$, and we can write $X_k ... X_1 P_t = X_k Q_{k-1} X_{k-1} Q_{k-2} ... X_1 Q_0$, where we have denoted P_t as Q_0. Then we get the following formula:

$$\begin{aligned}\det(X_k ... X_1 P_t) &= \det(X_k Q_{k-1}) ... \det(X_2 Q_1) \det(X_1 Q_0) \\ &= [\det(X_1 P_u)]^{k-1} \det(X_1 P_t).\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} n^{-1} \log\det(\Pi_n P_t) = \log\det(X_1 P_u).$$

QED.

4 Example

Let us compute the Lyapunov exponents for a random variable X that has the product X^*X distributed according to the Marchenko-Pastur distribution. Recall that the continuous part of the Marchenko-Pastur probability distribution with parameter $\lambda > 0$ is supported on the interval $\left[\left(1-\sqrt{\lambda}\right)^2, \left(1+\sqrt{\lambda}\right)^2\right]$, and has the following density there:

$$p_\lambda(x) = \frac{\sqrt{4\lambda - (x-1-\lambda)^2}}{2\pi x}.$$

For $\lambda \in (0,1)$, this distribution also has an atom at 0 with the probability mass $(1-\lambda)$ assigned to it. The Marchenko-Pastur distribution is sometimes called the free Poisson distribution since it arises as a limit of free additive convolutions of the Bernoulli distribution, and a similar limit in the classical case equals the Poisson distribution. It can also be thought of as a scaled limit of the eigenvalue distribution of Wishart-distributed random matrices (see [6] for a discussion).

Proposition 2 *Suppose that X is a non-commutative random variable in a tracial W^*-probability space $(\mathcal{A}, E)$, such that X^*X is distributed according to the Marchenko-Pastur distribution with parameter λ. If $\lambda \geq 1$, then the distribution of Lyapunov exponents of X is*

$$\mathcal{F}(x) = \begin{cases} 0, & \text{if } x < (1/2)\log(\lambda - 1) \\ e^{2x} + 1 - \lambda, & \text{if } x \in \left[\frac{1}{2}\log(\lambda-1), \frac{1}{2}\log\lambda\right) \\ 1 & \text{if } x \geq \frac{1}{2}\log\lambda. \end{cases}.$$

If $\lambda < 1$, then the distribution of Lyapunov exponents of X is

$$\mathcal{F}(x) = \begin{cases} e^{2x}, & \text{if } x < (1/2)\log(\lambda) \\ \lambda, & \text{if } x \in \left[\frac{1}{2}\log(\lambda), 0\right) \\ 1 & \text{if } x \geq 0. \end{cases}$$

Remark: If $\lambda = 1$, then the distribution is the exponential law discovered by C. M. Newman as a scaling limit of Lyapunov exponents of large random matrices. (See [9], [10], and [7]. This law is often called the "triangle" law since it implies that the exponentials of Lyapunov exponents converge to the law whose density function is in the form of a triangle.)

Proof of Proposition 2: It is easy to calculate that the continuous part of the distribution of P_tXP_t is supported on the interval $\left[\left(\sqrt{t}-\sqrt{\lambda}\right)^2, \left(\sqrt{t}+\sqrt{\lambda}\right)^2\right]$, and has the density function

$$p_{t,\lambda}(x) = \frac{\sqrt{4\lambda t - [x-(t+\lambda)]^2}}{2\pi x}.$$

This distribution also has an atom at $x = 0$ with the probability mass $\max\{1-\lambda, 1-t\}$. See for example, results in [12].

Next, we write the expression for the integrated Lyapunov exponent. If $\lambda \geq 1$, or $\lambda < 1$ but $\lambda \geq t$, then

$$\begin{aligned} F_\lambda(t) &= \frac{1}{2}\lim_{\varepsilon \downarrow 0} E \log^{+\varepsilon}(P_t X^* X P_t) \\ &= \frac{1}{2}\int_{\left(\sqrt{t}-\sqrt{\lambda}\right)^2}^{\left(\sqrt{t}+\sqrt{\lambda}\right)^2} \log x \frac{\sqrt{4\lambda t - [x-(t+\lambda)]^2}}{2\pi x} dx. \end{aligned} \tag{7}$$

If $\lambda < 1$ and $\lambda < t$, then

$$\begin{aligned} F_\lambda(t) &= \frac{1}{2}\lim_{\varepsilon \downarrow 0} E \log^{+\varepsilon}(P_{1-\lambda} X^* X P_{1-\lambda}) \\ F_\lambda(t) &= \frac{1}{2}\int_{\left(\sqrt{1-\lambda}-\sqrt{\lambda}\right)^2}^{\left(\sqrt{1-\lambda}+\sqrt{\lambda}\right)^2} \log x \frac{\sqrt{4\lambda(1-\lambda) - [x-1]^2}}{2\pi x} dx. \end{aligned} \tag{8}$$

Differentiating (7) with respect to t, we obtain an expression for the marginal Lyapunov exponent:

$$f_\lambda(t) = \frac{1}{4\pi}\int_{\left(\sqrt{t}-\sqrt{\lambda}\right)^2}^{\left(\sqrt{t}+\sqrt{\lambda}\right)^2} \frac{\log x}{x} \frac{x-t+\lambda}{\sqrt{4\lambda x - [x-t+\lambda]^2}} dx. \tag{9}$$

Using substitutions $u = \left[x - \left(\sqrt{t}-\sqrt{\lambda}\right)^2\right] / \left(2\sqrt{\lambda t}\right) - 1$ and then $\theta = \arccos u$, this integral can be computed as

$$f_\lambda(t) = \frac{1}{2}\log(\lambda - t).$$

From this expression, we calculate the distribution of Lyapunov exponents for the case when $\lambda \geq 1$:

$$\mathcal{F}(x) = \begin{cases} 0, & \text{if } x < (1/2)\log(\lambda-1) \\ e^{2x} + 1 - \lambda, & \text{if } x \in \left[\frac{1}{2}\log(\lambda-1), \frac{1}{2}\log\lambda\right) \\ 1 & \text{if } x \geq \frac{1}{2}\log\lambda. \end{cases}$$

A similar analysis shows that for $\lambda < 1$, the distribution is as follows:

$$\mathcal{F}(x) = \begin{cases} e^{2x}, & \text{if } x < (1/2)\log(\lambda) \\ \lambda, & \text{if } x \in \left[\frac{1}{2}\log(\lambda), 0\right) \\ 1 & \text{if } x \geq 0. \end{cases}$$

QED.

5 A relation with the S-transform

In this section we derive a formula that makes the calculation of Lyapunov exponents easier and relates them to the S-transform of the operator X_i. Recall that the ψ-function of a bounded non-negative operator A is defined as $\psi_A(z) = \sum_{k=1}^{\infty} E\left(A^k\right) z^k$. Then the S-transform is $S_A(z) = \left(1+z^{-1}\right)\psi_A^{(-1)}(z)$, where $\psi_A^{(-1)}(z)$ is the functional inverse of $\psi_A(z)$ in a neighborhood of 0.

Theorem 2 *Let X_i be identically distributed free bounded operators in a tracial W^*-probability space $\mathcal{A}$ with trace E. Let $Y = X_1^* X_1$ and suppose that $\mu_Y(\{0\}) = 1-u \in [0,1)$, where μ_Y denotes the spectral probability measure of Y. Then the marginal Lyapunov exponent of the sequence $\{X_i\}$ is given by the following formula:*

$$f_X(t) = \begin{cases} -\frac{1}{2}\log\left[S_Y(-t)\right] & \text{if } t < u, \\ 0 & \text{if } t > u, \end{cases}$$

where S_Y is the S-transform of the variable Y.

Remark: Note that if $X_1^* X_1$ has no atom at zero then the formula is simply $f_X(t) = -\frac{1}{2}\log\left[S_Y(-t)\right]$.

Proof: If $t > u$, then $f_X(t) = 0$ by Theorem 1. Assume in the following that $t < u$. Then $P_t X^* X P_t$ has an atom of mass $1-t$ at 0. Let μ_t denote the spectral probability measure of $P_t X^* X P_t$, with the atom at 0 removed. (So the total mass of μ_t is t.) We start with the formula:

$$\log x = \log(c+x) - \int_0^c \frac{ds}{x+s},$$

and write:

$$\int_0^{\infty} \log x \; \mu_t(dx) = \lim_{c\to\infty}\left\{t\log(c) + \int_0^c G_t(-s)\,ds\right\},$$

where G_t is the Cauchy transform of the measure μ_t.

Next, note that $G_t(-s) = -s^{-1}\psi_t\left(-s^{-1}\right) - ts^{-1}$ and substitute this into the previous equation:

$$\begin{aligned} \int_0^{\infty} \log x \; \mu_t(dx) &= \lim_{c\to\infty,\varepsilon\to 0}\left\{t\log c - t\log c + t\log(\varepsilon) + \int_{\varepsilon}^{c} \frac{\psi_t\left(s^{-1}\right)}{s} ds\right\} \\ &= \lim_{\varepsilon\to 0}\left\{t\log(\varepsilon) + \int_{\varepsilon}^{\infty} \frac{\psi_t\left(s^{-1}\right)}{s} ds\right\}. \end{aligned}$$

Using substitutions $v = -\log s$, and $A = -\log\varepsilon$, we can re-write this equation as follows:

$$\int_0^{\infty} \log x \; \mu_t(dx) = \lim_{A\to\infty}\left\{-tA - \int_{-\infty}^{A} \psi_t\left(-e^v\right) dv\right\}.$$

The function $\psi_t(-e^v)$ monotonically decreases when v changes from $-\infty$ to ∞, and its value changes from 0 to $-t$. Let $s^* =: \psi_t\left(-e^0\right) = \psi_t(-1)$ and let $\xi_t(x)$ denote the functional inverse of $\psi_t(-e^v)$. The function $\xi_t(x)$ is defined on the interval $(-t, 0)$. In this interval it is monotonically decreasing from ∞ to $-\infty$. The only zero of $\xi_t(x)$ is at $x = s^*$.

It is easy to see that

$$\lim_{A\to\infty}\left\{-tA - \int_0^A \psi_t(-e^v)\,dv\right\} = -\int_{-t}^{s^*} \xi_t(x)\,dx,$$

and that

$$-\int_{-\infty}^0 \psi_t(-e^v)\,dv = -\int_{s^*}^0 \xi_t(x)\,dx.$$

Therefore,

$$\int_0^\infty \log x\ \mu_t(dx) = -\int_{-t}^0 \xi_t(x)\,dx.$$

It remains to note that $\xi_t(x) = \log\left[-\psi_t^{(-1)}(x)\right]$, in order to conclude that

$$\int_0^\infty \log x\ \mu_t(dx) = -\int_{-t}^0 \log\left[-\psi_t^{(-1)}(x)\right] dx.$$

The next step is to use Voiculescu's multiplication theorem and write: $\psi_t^{(-1)}(x) = \psi_Y^{(-1)}(x)(1+x)/(t+x)$. Then we have the formula:

$$\begin{aligned}\int_0^\infty \log x\ \mu_t(dx) &= -\int_{-t}^0 \log\left[-\psi_Y^{(-1)}(x)\right] dx - \int_{-t}^0 \log\left[\frac{1+x}{t+x}\right] dx \\ &= -\int_{-t}^0 \log\left[-\psi_Y^{(-1)}(x)\right] dx + (1-t)\log(1-t) + t\log t.\end{aligned}$$

The integrated Lyapunov exponent is one half of this expression, and we can obtain the marginal Lyapunov exponent by differentiating over t:

$$\begin{aligned}f(t) &= \frac{1}{2}\left(-\log\left[-\psi_Y^{(-1)}(-t)\right] + \log t - \log(1-t)\right) \\ &= -\frac{1}{2}\log\left[\left(1-\frac{1}{t}\right)\psi_Y^{(-1)}(-t)\right] \\ &= -\frac{1}{2}\log\left[S_Y(-t)\right].\end{aligned}$$

QED.

Example

Let us consider again the case of identically distributed free X_i such that $X_i^* X_i$ has the Marchenko-Pastur distribution with the parameter $\lambda \geq 1$. In this

case $S_Y(z) = (\lambda + z)^{-1}$. Hence, applying Theorem 2, we immediately obtain a formula for the marginal Lyapunov exponent:

$$f(t) = \frac{1}{2}\log(\lambda - t).$$

Inverting this formula, we obtain the formula for the distribution of Lyapunov exponents:

$$\mathcal{F}(x) = \begin{cases} 0, & \text{if } x < (1/2)\log(\lambda - 1), \\ e^{2x} + 1 - \lambda, & \text{if } x \in \left[\frac{1}{2}\log(\lambda - 1), \frac{1}{2}\log\lambda\right), \\ 1 & \text{if } x \geq \frac{1}{2}\log\lambda, \end{cases}$$

which is exactly the formula that we obtained earlier by a direct calculation from definitions. It is easy to check that a similar agreement holds also for $\lambda < 1$.

In the following corollaries we always assume that operators belong to a tracial W^*-probability space $\mathcal{A}$ with trace E.

Corollary 1 *Let X and Y be free and such that X^*X and Y^*Y are bounded and have no atom at zero. Let f_X, f_Y, and f_{XY} denote the marginal Lyapunov exponents corresponding to variables X, Y and XY, respectively. Then*

$$f_{XY}(t) = f_X(t) + f_Y(t).$$

Proof: By Theorem 2,

$$\begin{aligned} f_{XY}(t) &= -\frac{1}{2}\log\left[S_{Y^*X^*XY}(-t)\right] \\ &= -\frac{1}{2}\log\left[S_{Y^*Y}(-t)\, S_{X^*X}(-t)\right] \\ &= f_X(t) + f_Y(t). \end{aligned}$$

QED.

Corollary 2 *If X is bounded and X^*X has no atom at zero, then the marginal Lyapunov exponent is (weakly) decreasing in t, i.e. $f_X'(t) \leq 0$.*

Proof: Because of Theorem 2, we need only to check that $S(t)$ is (weakly) decreasing on the interval $[-1, 0]$, and this was proved by Bercovici and Voiculescu in Proposition 3.1 on page 225 of [1]. QED.

Corollary 3 *If X is bounded and X^*X has no atom at zero, then the largest Lyapunov exponent equals $(1/2)\log E(X^*X)$.*

Proof: This follows from the previous Corollary and the fact that $S_Y(0) = 1/E(Y)$. QED.

Remark: It is interesting to compare this result with the result in [2], which shows that the norm of the product of $N \times N$ i.i.d. random matrices $X_1, \ldots, X_n$ grows exponentially when n increases, and that the asymptotic growth rate approaches $\frac{1}{2} \log E(\mathrm{tr}(X_1^* X_1))$ if $N \to \infty$ and matrices are scaled appropriately. The assumption in [2] about the distribution of matrix entries is that the distribution of $X_1^* X_1$ is invariant relative to orthogonal rotations of the ambient space. Since the growth rate of the norm of the product $X_1 \ldots X_n$ is another way to define the largest Lyapunov exponent of the sequence X_i, therefore the result in [2] is in agreement with Corollary 3.

The main result of Theorem 2 can also be reformulated as the following interesting identity:

Corollary 4 *If Y is bounded, self-adjoint, and positive, and if $\{P_t\}$ is a family of projections which are free of Y and such that $E(P_t) = t$, then*

$$\begin{aligned} \log S_Y(-t) &= -\frac{d}{dt}\left[\lim_{\lambda \to 0} E \log^{+\lambda}(P_t Y P_t)\right] \\ &= -2\frac{d}{dt}\left[\log \det\left(\sqrt{Y} P_t\right)\right]. \end{aligned}$$

Conversely, we can express the determinant in terms of the S-transform:

Corollary 5 *If X is bounded and invertible, then*

$$\det(X) = \exp\left\{-\frac{1}{2}\int_0^1 \log S_{X^*X}(-t)\, dt\right\}.$$

6 Conclusion

In conclusion we want to indicate how our results are related to results by Newman in [9]. Newman considers N-by-N random matrices X_i with rotationally-invariant distribution.of entries. Suppose that the empirical distribution of eigenvalues of $\sqrt{X_i^* X_i}$ converges to a probability measure $K(dx)$ as $N \to \infty$. Let $H(x)$ be the limit cumulative distribution function for e^{f_k}, where f_k are Lyapunov exponents of X_i. Newman shows that this limit exists and satisfies the following integral equation:

$$\int_0^\infty \frac{t^2}{H(x)\, x^2 + (1 - H(x))\, t^2} K(dt) = 1.$$

We are going to transform this expression in a form which is more comparable with our results. Let $\mu(dt)$ be the limit of the empirical distribution of

eigenvalues of $X_i^* X_i$ as $N \to \infty$, and let $s = t^2$. Then, we can re-write Newman's formula as follows:

$$\int_0^\infty \frac{s}{H(x)\,x^2 + (1 - H(x))\,s} \mu(ds) = 1.$$

This expression can be transformed to the following equality:

$$\psi_Y\left(\frac{H(x) - 1}{H(x)\,x^2}\right) = H(x) - 1,$$

where $\psi_Y(z)$ is the ψ-function of a non-commutative random variable Y with the spectral probability distribution μ. Hence

$$\frac{1}{x^2} = S_Y[H(x) - 1].$$

If we denote $1 - H(x)$ as u, then

$$x = e^{-\frac{1}{2}\log[S_Y(-u)]}.$$

This implies that as N grows, the ordered Lyapunov exponents f_k converge to $-\frac{1}{2}\log[S_Y(-u)]$, provided that $k/N \to u$. This is in agreement with our formula in Theorem 2.

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